

Q. 1) Solve: $xp + yq = z$.

Solution: - Comparing the given equation to the Lagrange's linear d. e. $Pp + Qq = R$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

Here, $P = x$, $Q = y$ and $R = z$

Then Lagrange's auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

From first two fractions, we have

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating,

$$\log x = \log y + \log c_1 \Rightarrow x = c_1 y$$

$$\therefore c_1 = \frac{x}{y}$$

Again taking the last two fractions, we have

$$\frac{dy}{y} = \frac{dz}{z} \text{ and integrating}$$

$$\log y = \log z + \log c_2 \quad \therefore c_2 = \frac{y}{z}$$

Hence the general solution of the given equation is

$$\Phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$$

Q.2) solve: $(mz - ny)p + (nx - lz)q = ly - mx$

Solution: — Here, Lagrange's auxiliary eqns. are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Now, choosing x, y, z as multipliers, then each fraction = $x dx + y dy + z dz$

$$\frac{x(mz - ny) + y(nx - lz) + z(ly - mx)}{x(mz - ny) + y(nx - lz) + z(ly - mx)} = 0$$

$$\therefore x dx + y dy + z dz = 0$$

$$\text{Integrating } x^2 + y^2 + z^2 = C_1 \dots \dots \dots (1)$$

Again choosing l, m, n as multipliers, then

$$\text{each fraction} = \frac{l dx + m dy + n dz}{\dots}$$

$$\frac{\sum l(mz - ny)}{\dots} = 0$$

$$\therefore l dx + m dy + n dz = 0$$

Integrating, we get

$$lx + my + nz = C_2 \dots \dots \dots (2)$$

By using ~~from~~ (1) and (2), we get general solution of the given equation.

$$\text{i.e. } \phi(x^2 + y^2 + z^2, lx + my + nz) = 0$$

Q.3) solve: $(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$

Solution: —

$$\text{Here, a.e. are } \frac{dx}{y^2 + z^2 - x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$$

Taking the last two fractions, we have

$$\frac{dy}{-2xy} = \frac{dz}{-2xz} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

$$\text{Integrating, } \log y = \log z + \log C_1 \quad \therefore \boxed{C_1 = \frac{y}{z}}$$

Now, choosing x, y, z as multipliers, we get ⁶⁶

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz} \text{ (each fraction)}$$

$$= \frac{xdx + ydz + zdz}{x^2y + xz^2 - x^3 - 2xy^2 - 2xz^2} = \frac{xdx + ydy + zdz}{-xy^2 - xz^2 - x^3}$$

$$= \frac{xdx + ydy + zdz}{-x(x^2 + y^2 + z^2)}$$

From the last two fractions, we have

$$\frac{dz}{+2xz} = \frac{xdx + ydy + zdz}{+x(x^2 + y^2 + z^2)}$$

Integrating, we get

$$\frac{1}{2} \log z = \frac{1}{2} \log(x^2 + y^2 + z^2) - \frac{1}{2} \log C_2$$

$$\therefore zC_2 = x^2 + y^2 + z^2$$

$$\therefore C_2 = \frac{x^2 + y^2 + z^2}{z}$$

Hence $\frac{x^2 + y^2 + z^2}{z} = \phi\left(\frac{y}{z}\right)$

which is the general solution.

Q. (A) Solve $p + 3q = 5z + \tan(y - 3x)$

Solution: - Here a. eqns. are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}$$

Taking the first two fractions, we have

$$\frac{dx}{1} = \frac{dy}{3} \Rightarrow 3dx = dy$$

Integrating, $3x = y + C_1$

$$\therefore -C_1 = -3x - y$$

$$\Rightarrow C_1 = y - 3x \quad \text{--- (1)}$$

Again taking the first and the last fractions, we have

$$\frac{dx}{1} = \frac{dz}{5z + \tan(\gamma - 3x)} = \frac{dz}{5z + \tan \gamma} \quad [\text{from (1)}]$$

Integrating,

$$x = \frac{1}{5} \log(5z + \tan \gamma) + \frac{1}{5} C_2 \quad (\text{we can take})$$

$$\Rightarrow C_2 = 5x - \log(5z + \tan \gamma)$$

$$\Rightarrow C_2 = 5x - \log[5z + \tan(\gamma - 3x)] \quad \text{--- (2)}$$

From (1) and (2), we can find the general solution of the given equation which is

$$5x - \log[5z + \tan(\gamma - 3x)] = f(\gamma - 3)$$

Q.15) Solve: $z - px - qy = a(x^2 + y^2 + z^2)^{\frac{1}{2}}$

Solution: — The given equation can be written in its standard form $(Px + Qy = R)$ as

$$px + qy = z - a(x^2 + y^2 + z^2)^{\frac{1}{2}}$$

The a. eqns. are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2 + y^2 + z^2}}$$

Taking the first two fractions and after solving

$$\text{we get } C_1 = \frac{x}{y} \quad \text{--- (1)}$$

Again taking x, y, z as multipliers,

$$\text{each fraction} = \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{x dx + y dy + z dz}{x^2 + y^2 + z^2 - a z \sqrt{x^2 + y^2 + z^2}} \quad \text{--- (2)}$$

Putting $x^2 + y^2 + z^2 = t^2$ so that

Equation (2) becomes

$$\frac{dz}{z-at} \quad x dx + y dy + z dz = t dt$$

Now, eqn. (2) becomes

$$\frac{dz}{z-at} = \frac{t dt}{t^2 - a z t} \Rightarrow \frac{dz}{z-at} = \frac{dt}{t-az}$$

Thus $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z-at} = \frac{dt}{t-az} = \frac{dz + dt}{z-at + t-az} = \frac{dz + dt}{(1-a) \cdot (t+z)}$

Taking the first and the last fractions, we have

$$\frac{dx}{x} = \frac{dz + dt}{(1-a)(t+z)} \Rightarrow \log x = \frac{1}{(1-a)} \cdot \log(t+z) + \log c_2$$

$$\Rightarrow \log x^{1-a} = \log(t+z) + \log c_2$$

[∵ $\log c_2$ and $(1-a) \log c_2$ both are constants]

$$\therefore c_2 = \frac{x^{1-a}}{t+z} = \frac{x^{1-a}}{z + \sqrt{x^2 + y^2 + z^2}} \quad (3)$$

From (1) and (3) we find the required solution which is

$$F\left(\frac{x}{y}, \frac{x^{1-a}}{z + \sqrt{x^2 + y^2 + z^2}}\right) = 0$$

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